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Adaptive Topology Optimization of Shell Structures

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To solve generalized shape optimization problems, material topology optimization is applied to determine the basic layout, i.e., to insert and eliminate holes in the design space. In addition, the inner and outer contours are optimized in detail by shape optimization. To improve the quality of the results and to increase the numerical efficiency, the design model is adapted during the optimization process. Two aspects of design adaptivity are considered. On the one hand, the parametrization of the design space is varied during the material topology optimization process and, on the other hand, the kind of design model is adapted during the overall optimization process switching between material topology and shape optimization. The adaptive techniques are discussed, and adaptive design modeling is verified for topology optimization problems of shell structures.

Introduction

STRUCTURAL optimization can be split into three subproblems: 1) modeling of the structural layout, 2) analyzing the structural response and the sensitivities of the structural response with respect to a variation of the layout, and 3) generating an improved layout. In general, these subproblems cannot be solved explicitly. Therefore, the design function and the state variables have to be discretized. The design function describes the structural layout, e.g., by the shape. The state variables describe the structural response, e.g., by the stress distribution. The resulting parametric problems are solved numerically: the design variables can be determined by either mathematical programming or optimality criteria methods; the discretized state variables can be determined, for example, by the finite element method.

Conventionally, at the beginning of the optimization process the kind and degree of discretization of both the design function and the state variables are fixed. However, during the optimization process the structural layout often changes considerably and the initial discretizations approximate the real optimum design and the related structural response only very roughly. This holds, in particular, for topology optimization problems, where the initial design and the optimum layout differ considerably. Consequently, adaptive procedures are necessary to adapt the discretizations in the design and in the analysis model to the current state of the optimization process.

Within the framework of structural optimization the adaptation of the finite element mesh was mainly investigated in recent years. To control the quality of the finite element analysis, Bennet and Botkin[1] introduced adaptive mesh refinement in shape optimization. Based on the strain energy distribution, the number of elements was increased. This idea was also proposed by Kikuchi et al.,[2] applying a Z_2 type error indicator in combination with h- and r-refinement strategies. To consider the variation of the design, as well as to improve the quality of the sensitivities, Banichuk et al.[3] additionally refined elements along variable boundaries. More rigorous methods to consider the error in the sensitivities of the state variables were published by Barthold et al.[4] and Maute and Ramm,[5] applying an extended Z_2 error indicator, and by Buscaglia et al.,[6] applying a residual error estimate. However, only a few schemes have been described to adapt the discretization of the design function. The advantages of design adaptivity in shape optimization were verified by Mathiak and Schnack,[7] refining the discretization of boundary segments with large curvature. Alternatively, Falk et al.[8] introduced a hierarchical design element concept. The optimization variables, i.e., the position of the control nodes of splines, are assigned to different levels depending on the influence of the optimization variable on the shape. First, only a few variables on the top level with a global influence on the overall shape are varied. Variables of lower levels are linked to these so-called master variables. Based on this geometrical discretization, the optimum shape is determined. In the following steps, variables of the lower levels are selected to become independent optimization variables. The refinement criteria are based on the derivatives of the Lagrangian function of the additional variables. This procedure increases the numerical efficiency of the optimization process and enables the user to restrict the degree of refinement in advance.

The benefits of an adaptive approach in material topology optimization for plane structures were presented by Maute and Ramm.[9] In the present paper, design adaptivity is discussed within the framework of topology optimization in the context of generalized shape optimization of shell structures. An overall procedure is presented that allows the adapting of the design parametrization during the optimization process. How a new adapted design model can be generated and how data are mapped from one design model to the other are outlined. Two different aspects of design adaptivity are considered. To begin, a specific adaptive procedure in material topology optimization of shell structures is presented, and then the combination of adaptive material topology and shape optimization is discussed. The proposed procedures are verified for shell problems.

Design Modeling and Design Adaptivity

In principle, there are two possibilities to describe the structural layout and its variation in the optimization process: conventional shape optimization and material topology optimization. In the first case the geometry of a structure is defined by contours and surfaces assuming that the enclosed area or volume is continuously filled with material. The main advantage of this approach is that only the boundaries have to be parametrized by shape functions to solve the geometrical problem numerically. This leads to rather few optimization variables, i.e., the control nodes of the shape functions, which define the initial geometry as well as the optimized geometry of the structure. However, new boundaries cannot emerge during the optimization process. Only contours or surfaces that exist in the initial layout can be varied, no topology change is allowed.

To vary the conceptual layout of a structure, a discontinuous approach is necessary. In material topology optimization the body $\boldsymbol{X}$ of a structure is defined, whether or not there is material at a point $\boldsymbol{x}$ in the design space Ω (Ref. 10):

$$\boldsymbol{X}\colon \chi(\boldsymbol{x}) = \begin{cases} 0 \rightarrow \text{no material} \\ 1 \rightarrow \text{material} \end{cases} \qquad \boldsymbol{x} \in \Omega \tag{1}$$

In this scheme only one homogeneous, isotropic material in the design space is assumed. This description is not based on a continuous material distribution, i.e., arbitrary material layouts are possible allowing the change in the topology. However, the material distribution has to be discretized by so-called design patches with a high

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resolution to identify clearly the optimum structural layout. In general, this pixellike approach leads to extremely large numbers of optimization variables, which are the parameters of the discretized material distribution. On the other side, the clearness of the optimization results is lost if only a coarse parametrization of the design space is used.

The advantages of the geometrical approach can be used in topology optimization and the disadvantages of material topology optimization can be avoided if an adaptive design model is used. On a higher level of adaptivity, the kind of design model is changed. Material topology is applied to generate or eliminate holes, i.e., to change the topology, and the shape of inner and outer boundaries is determined in detail by pure shape optimization. Shape and material topology optimization can be applied alternately or simultaneously. On a lower level, the quality of the results and the numerical efficiency in material topology optimization can be improved by adapting the discretization of the material distribution in the design space during the optimization process. On the same level, shape optimization can be improved by adaptive techniques, too. However, this is not considered in this study; design adaptivity in shape optimization is described, e.g., in Ref. 11.

Adaptive Optimization Process

Regardless of the level of design adaptivity the optimization process can be split into three steps.

1) Solve the optimization problem for a given parametrization by material topology and/or shape optimization.

2) Store the optimization results in a background mesh and determine the kind of design model and the degree of parametrization in the subsequent design model.

3) Update the design model, defining the new set of optimization variables and generating a new analysis model.

In addition to a conventional optimization process, a background mesh is introduced, which serves to store information when the design model is changed. A material description is used for this, which allows big shape changes and distinct topology variations. In the background mesh the body of the structure is parametrized by design patches with a very high resolution. Based on the material distribution, an adapted design model is generated, depending on the kind of optimization procedure to be applied (Fig. 1). A design model can be obtained for pure shape, pure topology, or combined topology/shape optimization. This process is discussed in the following sections. Regardless of what kind of optimization

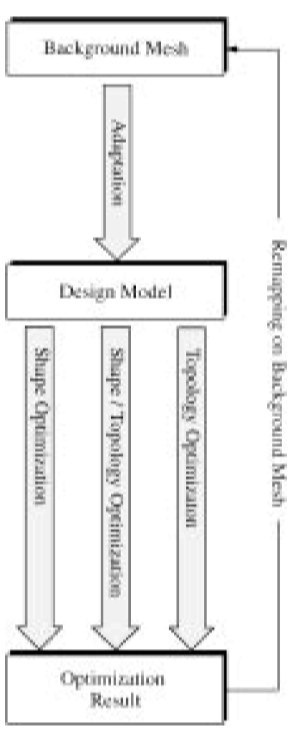


Fig. 1 Adaptive optimization process.

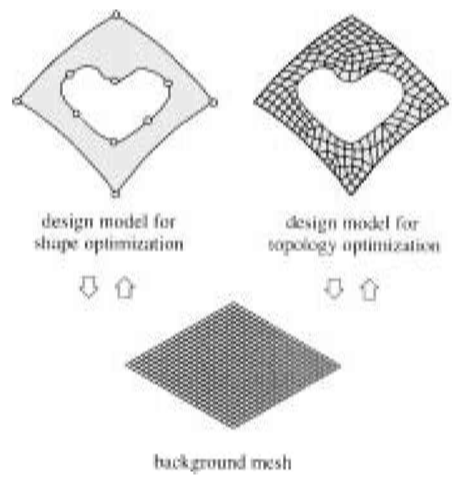


Fig. 2 Design model–background mesh.

procedure is applied, the optimization result is mapped back onto the background mesh (Fig. 2). From there, the updated material distribution is processed, and a new design model is automatically generated. This adaptive loop (called cycle) is repeated until convergence is reached. The convergence criterion of the overall procedure is the rate of objective or the change of the material distribution in the background mesh.

To make this optimization procedure feasible for the user, the generation of the design model and the following analysis model, the mapping and remapping of data from one model to the other and the optimization steps have to be automated to a great extent. In essence, the user has to choose only the kind of structural optimization in the following cycle. This decision is determined by the current stage of the overall optimization process, i.e., the clearness of the structural layout in conjunction with the different features of shape and topology optimization. However, it is important to provide an additional interface, where the user can modify the optimization problem, e.g., adding or removing optimization variables.

Modeling in Parameter Space

Considering shell optimization problems in three dimensions, it is advisable to model the structure not in the physical space $\boldsymbol{X} \in \mathbb{R}^3$ but in a two-dimensional parameter space $\boldsymbol{\zeta} \in \mathbb{R}^2$,

$$\boldsymbol{X}\colon \boldsymbol{\zeta} \rightarrow \boldsymbol{X}(\boldsymbol{\zeta}), \qquad \mathbb{R}^2 \rightarrow \mathbb{R}^3 \tag{2}$$

The background mesh, as well as the old and new design models, is defined in the parameter space. The relations between physical and parameter space are only effected by two basic operations: mapping a vector $\boldsymbol{q}$ from one onto the other space and vice versa.

A smooth projection $\boldsymbol{\zeta} \rightarrow \boldsymbol{X}(\boldsymbol{\zeta})$ is assumed. If $\boldsymbol{q} = \boldsymbol{q}(\boldsymbol{\zeta})$ is known, $\boldsymbol{q} = \boldsymbol{q}(\boldsymbol{X})$ can be determined:

$$\boldsymbol{q}(\boldsymbol{X}) = \boldsymbol{q}(\boldsymbol{\zeta}) \tag{3}$$

and

$$\frac{\mathrm{d}\boldsymbol{q}}{\mathrm{d}\boldsymbol{X}} = \boldsymbol{J}(\boldsymbol{\zeta})^{-1}\frac{\mathrm{d}\boldsymbol{q}}{\mathrm{d}\zeta} \tag{4}$$

with

$$J_{ij}(\boldsymbol{\zeta}) = \frac{\partial X_i}{\partial \zeta_j} \tag{5}$$

where J_{ij} denotes the Jacobian. Mapping a specific vector $\tilde{\boldsymbol{X}}$ in the physical space onto the parameter space is more costly because the projection $\boldsymbol{\zeta} \rightarrow \boldsymbol{X}(\boldsymbol{\zeta})$ is, in general, nonlinear. One possibility is to formulate a least square problem in $\boldsymbol{\zeta}$,

$$\|\tilde{\boldsymbol{X}} - \boldsymbol{X}(\boldsymbol{\zeta})\| \rightarrow \min \tag{6}$$

where $\|\cdot\|$ denotes the Euclidean norm. This problem can be solved efficiently by a Newton method using the second derivatives $\partial^2\boldsymbol{X}/\partial\boldsymbol{\zeta}^2$. If the material distribution needs to be mapped, the

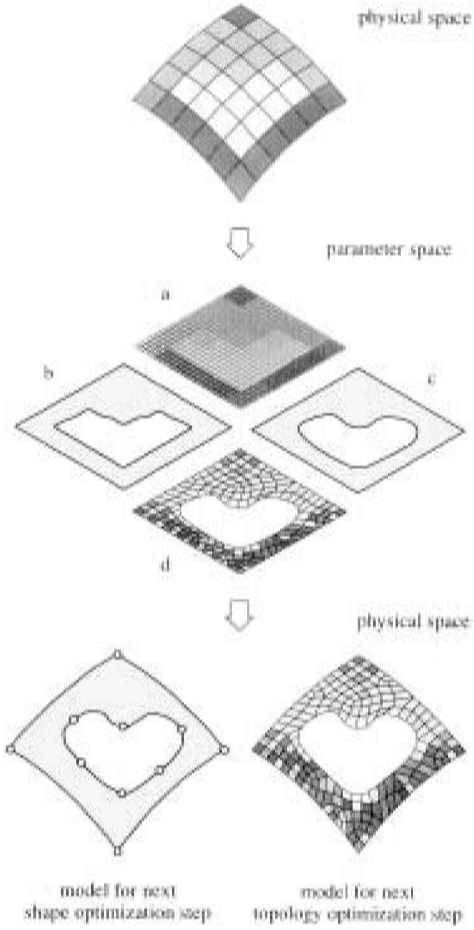


Fig. 3 Building a new design model.

relation between the areas Ω_ζ in the parameter and Ω_x in the physical space, respectively, also has to be determined:

$$\Omega_x = \int_{\zeta 1}\int_{\zeta 2} \det |\boldsymbol{F}|^{\frac{1}{2}}\, \mathrm{d}\Omega_\zeta \tag{7}$$

with

$$\boldsymbol{F} = \boldsymbol{J}^T \boldsymbol{J} \tag{8}$$

Generation of New Design Model

To generate a new design model due to the material distribution in the design space, extended techniques of image processing are used. The proposed algorithm is based on the following requirements.

1) The generated design model can be used for both shape and topology optimization.

2) The new design model contains only smooth boundaries, which are defined explicitly by shape functions. This allows a direct link to shape optimization.

3) The result of a previous optimization step can be modified, e.g., smoothed, but also directly transferred onto the new design model.

The fundamental idea of the proposed algorithm is to generate isolines of the material distribution represented by the density variation. These isolines can be used to define the inner and outer contours of the new design model. After the material distribution is mapped onto the background mesh in the parameter space, it can be optionally filtered, i.e., the material distribution can be lumped or smoothed. The isolines for one or more density levels are determined. For this the elemental material data of the design patches are transferred to nodal data. The points of one isoline are determined on the boundaries of the design patches by linear interpolation of the nodal density values. Connecting these points by polygons, the isolines are obtained. To get smooth boundaries the polygons are approximated by splines. The approximation tolerance, i.e., the area between the splines and the polygons, can be prescribed. The approximation error can be reduced by increasing the number or the order of splines. Because an adaptive approximation procedure is used, the segmentation by splines can be optimized. Domains with density values below a certain threshold value are considered as voids and, therefore, are neglected in the following optimization step. Finally, the generated structural layout is parametrized and the optimization variables are defined depending on the kind of optimization applied next, i.e., pure shape and/or topology optimization. Further details are given in Ref. 12.

The procedure is schematically shown in Fig. 3. The material distribution (densities) on a segment of a spherical shell is given. The material distribution is mapped from the analysis model in the physical space onto a square background mesh of 900 square patches in the parameter space (a). To determine the contours of the new design model, the isolines for a specific density ratio are approximated by polygons (b) and are smoothed by splines (c). Neglecting void domains, the remaining area is meshed and the material distribution stored in the background mesh is mapped onto the new mesh (d). Based on this mesh, a new design model is finally generated in the physical space for the next shape and topology optimization step.

Design Adaptivity in Material Topology Optimization

The main drawbacks of conventional material topology optimization, i.e., a large number of optimization variables and nonsmooth indefinite results, have their roots in a lack of flexibility. The parametrization of the overall design space is fixed during the entire optimization process. This problem can be overcome by an adaptive strategy discussed before. Within this strategy, called adaptive topology optimization[12] (ATO), it is possible to adapt the design model to the current material distribution. Thus, the number of optimization variables, i.e., the numerical effort, is reduced, and at the same time the quality of the optimization results is increased. Neglecting voids, the optimization process is confined to areas that are relevant for the structural layout. This does not lead to an irreversible process because the background mesh always spans the entire design space allowing the design model to shrink and grow. Furthermore, the parametrization can be controlled adaptively in nonvoid areas. In zones where the structural layout is not clear, the parametrization is refined by a free mesh generator. The iterative procedure leads directly to a smooth result using pure material topology optimization because the current structural layout in each cycle is approximated by smooth boundaries.

Maximum Stiffness Problems for Shell Structures

The advantages of design adaptivity are discussed for maximum stiffness problems of shell structures with mass constraint. These problems are frequently discussed in topology optimization.[13] The process is based on an artificial μ-powered material approach[14,15]:

$$\boldsymbol{C}(\rho) = \chi^\mu \boldsymbol{C}_0 \tag{9}$$

with

$$\chi = (\rho/\rho_0) \tag{10}$$

where $\boldsymbol{C}_0$ is the material tensor, ρ_0 the density of the homogeneous material, and ρ the variable material density. By a heuristically determined exponent $\mu > 1$, the intermediate density values are implicitly penalized and, therefore, the optimum design space is more or less clearly subdivided into void and solid areas. Thus, the optimization problem is formulated in a relaxed way,[16] i.e., $0 \le \rho \le \rho_0 \rightarrow 0 \le \chi \le 1$, as the stationary point of the corresponding Lagrangian function L,

$$L(\rho, \gamma) = \int_V \boldsymbol{\epsilon}^T \boldsymbol{C}\boldsymbol{\epsilon}\, \mathrm{d}v + \gamma \left[\int_V \rho\, \mathrm{d}v - \hat{m}\right] \tag{11}$$

where the product $\boldsymbol{\epsilon}^T\boldsymbol{C}\boldsymbol{\epsilon}$ yields the strain energy density and $\hat{m}$ is the prescribed design mass. The material distribution ρ is parametrized by piecewise constant shape functions ρ_i. The variations of the Lagrangian function with respect to the design variables ρ_i and the

Lagrange multiplier γ yield the Kuhn–Tucker conditions, which define the optimal solution

$$\frac{\partial L}{\partial \rho_i} = 0 \rightarrow \frac{\partial}{\partial \rho_i} \int_{V_i} \boldsymbol{\epsilon}^T \boldsymbol{C} \boldsymbol{\epsilon} \, \mathrm{d}v + \gamma V_i = 0 \tag{12}$$

$$\frac{\partial L}{\partial \gamma} = 0 \rightarrow \int_V \rho \, \mathrm{d}v - \hat{m} = 0 \tag{13}$$

where V_i is the volume of the related design patch. If the shape of the structure is not varied during the topology optimization process and the applied forces do not depend on the design variables, the following expression can be derived considering the principle of virtual displacement:

$$\frac{\partial(\boldsymbol{\epsilon}^T \boldsymbol{C} \boldsymbol{\epsilon})}{\partial \rho_i} = -\boldsymbol{\epsilon}^T \frac{\partial \boldsymbol{C}}{\partial \rho_i} \boldsymbol{\epsilon} \tag{14}$$

Based on the artificial approach according to Eq. (9), the derivation of $\boldsymbol{C}$ with respect to the optimization variable ρ_i together with Eqs. (12) and (14) leads to following optimality criterion:

$$-\frac{\mu}{\rho_i} \int_{V_i} \boldsymbol{\epsilon}^T \boldsymbol{C} \boldsymbol{\epsilon} \, \mathrm{d}v + \gamma V_i = 0 \tag{15}$$

Based on this optimality criterion, diverse optimization methods can evolve. The following recursive algorithm is applied:

$$\rho_i^{(n+1)} = \rho_i^{(n)} \left[\frac{\mu}{\gamma m_i} \int_{V_i} \boldsymbol{\epsilon}^T \boldsymbol{C} \boldsymbol{\epsilon} \, \mathrm{d}v_i \right]^{(n)\beta} \tag{16}$$

where (n) is the current iteration and m_i the mass in the design patch i. The exponent β is introduced to control the rate of convergence of the optimization process. If β is too large, oscillations may occur; if β is too small, the rate of convergence is low. Usually, β is chosen in the range between 0.6 and 0.8. The Lagrange multiplier γ is determined from Eq. (13) such that the mass constraint is satisfied.

In shell problems it is often useful to distinguish between strain energies due to bending e_b, membrane forces e_m, and transverse shear forces e_s and to consider their individual contributions in Eq. (16). In particular for thin shells, if shear locking may occur, the transverse shear energy leads to incorrect results, i.e., checkerboards. Based on the Reissner–Mindlin assumptions, the in-plane stresses σ_{11}, σ_{22}, and σ_{12} and strains $\varepsilon_{11}, \varepsilon_{22}$, and ε_{12} (Fig. 4) can be split into a uniform part and a linear part in thickness direction $\zeta_3 = [-1; 1]$, e.g., for σ_{12},

$$\sigma_{12} = \sigma_{12}^0 + (\zeta_3/2)\sigma_{12}^1 \tag{17}$$

The distribution of the transverse shear stresses σ_{13} and σ_{23} and strains ε_{13} and ε_{23} is assumed to be parabolic, e.g., σ_{13},

$$\sigma_{13} = \tfrac{3}{2}\sigma_{13}^0\left(1 - \zeta_3^2\right) \tag{18}$$

The integral over the thickness h of the shell

$$e = \frac{h}{2} \int_{-1}^{1} \sigma_{ij} \epsilon_{ij} \, \mathrm{d}\zeta_3 \tag{19}$$

yields the strain energy densities due to bending, membrane effects, and shear forces

$$e_m = (h/2)\left(\sigma_{11}^0 \epsilon_{11}^0 + \sigma_{22}^0 \epsilon_{22}^0 + \sigma_{12}^0 \epsilon_{12}^0\right) \tag{20}$$

$$e_b = (h^3/24)\left(\sigma_{11}^1 \epsilon_{11}^1 + \sigma_{22}^1 \epsilon_{22}^1 + \sigma_{12}^1 \epsilon_{12}^1\right) \tag{21}$$

$$e_s = \tfrac{3}{5} h\left(\sigma_{13}^0 \epsilon_{13}^0 + \sigma_{23}^0 \epsilon_{23}^0\right) \tag{22}$$

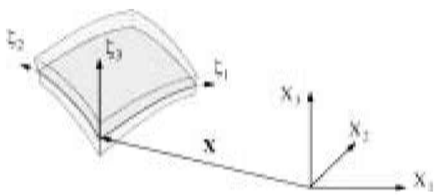


Fig. 4 Local and global coordinates.

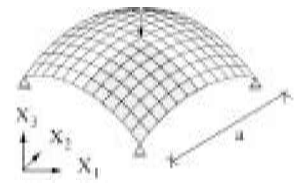


Fig. 5 Spherical shell subjected to a single load: diameter of sphere *d*, 20.00; length *a*, 5.00; and thickness *h*, 0.05.

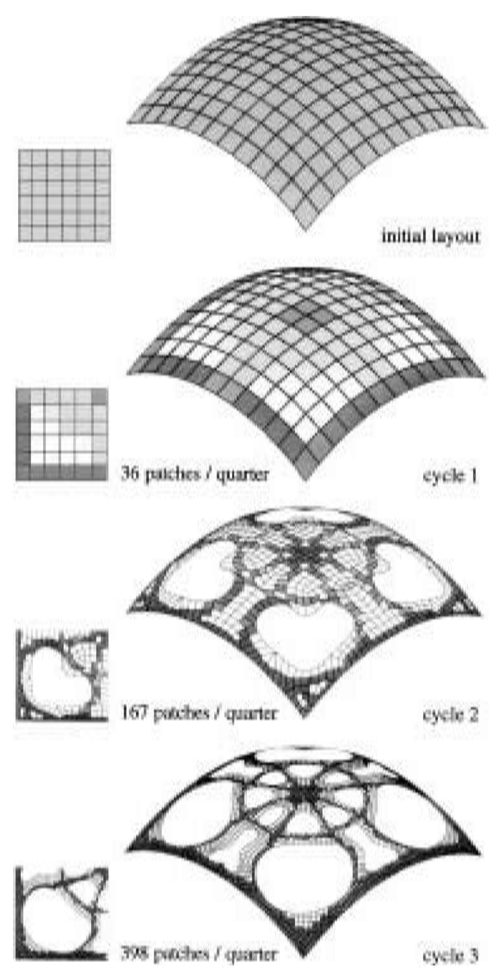


Fig. 6 Adaptive material topology optimization.

Adaptive Topology Optimization of a Spherical Shell

In Fig. 5 a shell example is shown. A segment of a spherical shell is subjected to a concentrated load in the vertex and vertically supported in the four corners. Because of the symmetry of the problem, only one-quarter of the shell is analyzed and optimized. The objective is maximum stiffness. The mass is kept constant during the optimization process. The density ratio ρ/ρ_0 in the initial homogeneous design space is 0.3. The μ exponent of the artificial material is 4.0. The material distribution problems of each cycle are solved by Eq. (16) considering the total strain energy.

The parametrizations of the design models together with the optimized material distributions for each optimization cycle are shown in Fig. 6. The number of design patches in each cycle is given. The iteration history of the objective with respect to the strain energy of the initial design (Fig. 7) shows that the objective is gradually decreased during the refinement of the design model. The peaks result from the mesh changes. To compare the adaptive method with the conventional procedure, the design problem was also solved in 32 iterations using one fixed fine mesh (900 patches per quarter) during the entire optimization process. The final layouts obtained by the adaptive scheme and the conventional nonadaptive procedure are shown in Fig. 8.

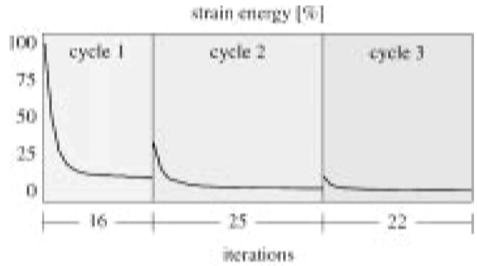


Fig. 7 Iteration history of adaptive optimization process.

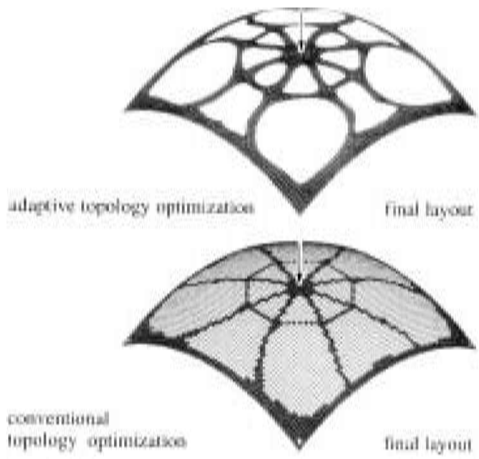


Fig. 8 Optimization results of adaptive and conventional procedure.

The overall numerical effort of the procedure can be approximated by

$$\text{overall effort} \sim \sum_{\text{cycle}} \text{effort for one analysis} \times \text{number of iterations} \tag{23}$$

Applying a direct solver, the effort for one linear elastic finite element analysis is approximately proportional to the power 2 of the number of degrees of freedom. It is assumed that the number of degrees of freedom is proportional to the number of finite elements, i.e., design patches,

$$\text{overall effort} \sim \sum_{\text{cycle}} (\text{number of elements})^2 \times \text{number of iterations} \tag{24}$$

Based on Eq. (24) it can be estimated that the overall numerical effort can be reduced by the adaptive method for this example to 16% in comparison to the conventional procedure. Moreover, applying adaptive techniques, a clearly defined structural layout results. However, despite a distinct improvement, the optimal shape of the outer and inner boundaries cannot be determined in detail by adaptive material topology optimization alone because an even finer parametrization along the boundaries would be necessary. Therefore, it is appropriate also to include shape optimization in the overall optimization procedure.

Combined Topology/Shape Optimization

To solve combined topology/shape optimization problems, it is not useful to combine only conventional material topology optimization methods with standard boundary variation techniques. In such a case only boundaries already existing in the initial design can be smoothed.[17] Another possibility is to use material topology as a preprocessor defining the conceptual layout and then to determine the detailed shape with a final shape optimization step. Based on the result of the topology optimization step, a geometric design model for shape optimization is generated. This can be done interactively[18] or automatically by image processing.[19] However, this procedure leads to considerable drawbacks. Because shape optimization is simply added to topology optimization, the basic layout of the structure cannot be changed any more once the topology is fixed. During the final shape optimization step, only boundaries defined in the related model can be varied. Consequently, neither can new holes be generated nor can existing holes be eliminated. However, because the topology of a structure influences the shape and vice versa, it is often not possible to find the attainable optimum. To overcome this drawback, material topology and shape optimization can be adaptively applied in parallel or alternately.

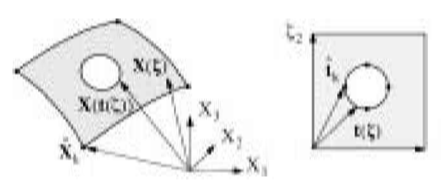
Shape Optimization on Curved Surfaces

In the present study shape optimization is applied to optimize contours defined on curved surfaces. Based on the design element concept,[20] the surface is parametrized in segments by shape functions N^k, e.g., Lagrangian or Bézier patches,[21]

$$\boldsymbol{X}(\boldsymbol{\zeta}) = \sum_k N^k(\boldsymbol{\zeta})\hat{\boldsymbol{X}}_k \tag{25}$$

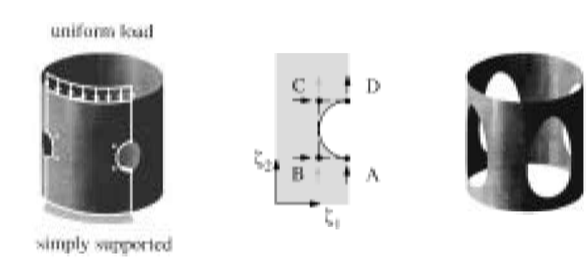
where $\hat{\boldsymbol{X}}_k$ are the position vectors of the control nodes. The main requirement of the design element concept is that a coarse mesh is possible to subdivide the initial structure into design patches. However, for complex topologies a large number of design patches may be necessary. This problem can be avoided by resorting to a hierarchical approach. The surface $\boldsymbol{X}(\boldsymbol{\zeta})$ is conventionally approximated by few design patches, ignoring the shape of inner and outer boundaries in the surface (Fig. 9). The boundaries $\boldsymbol{X}(\boldsymbol{t})$ are parametrized without a restrictive element concept by one-dimensional shape functions in the parameter space defined by the vector $\boldsymbol{\zeta}$. The problem of mapping of scalar and vector data defined in parameter space, such as the boundaries $\boldsymbol{t}(\boldsymbol{\zeta})$ and their derivatives $\partial\boldsymbol{t}/\partial\boldsymbol{\zeta}$, onto the physical space and vice versa was already outlined. Based on this hierarchical approach, two kinds of control nodes, i.e., optimization variables, have to be distinguished. One set of control nodes $\hat{\boldsymbol{X}}_k$ defines the overall shape of the surface within a design element approach; the other set of control nodes $\hat{\boldsymbol{t}}_k$ describes the shape of the boundaries on the surface within the design element. Only the latter is considered for topology shape optimization in this study.

In Fig. 10 this approach is used to determine the optimum shape of holes on a cylindrical surface. All holes are of the same shape. The objective is a homogeneous stress distribution considering von Mises stresses on the midsurface of the shell. Because of the multiple symmetry of the problem, only one-quarter of the structure is analyzed and optimized. The load and support conditions are shown in Fig. 10a. The shape of one-half of one hole is parametrized by one Bézier spline in the parameter space (Fig. 10b). The independent design variables are the ζ_2 positions of the control nodes A and D and the ζ_1 positions of the control nodes B and C. The ζ_2 positions of the nodes A and B, as well as C and D, are linked. One-quarter of the shell is discretized by eight-node isoparametric shell elements by a free mesh generator. Based on discrete analytical sensitivity



Shell surface **Parameter space**

Fig. 9 Hierarchical modeling.



a) Initial structure **b) Parameter space** **c) Optimized structure**

Fig. 10 Shape optimization on three-dimensional curved surfaces.

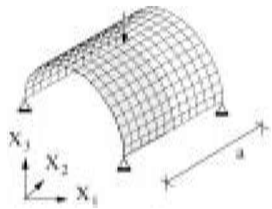


Fig. 11 Cylindrical shell: △, fixed; diameter of cylinder *d*, 2.00; length *a*, 2.00; and thickness *h*, 0.05.

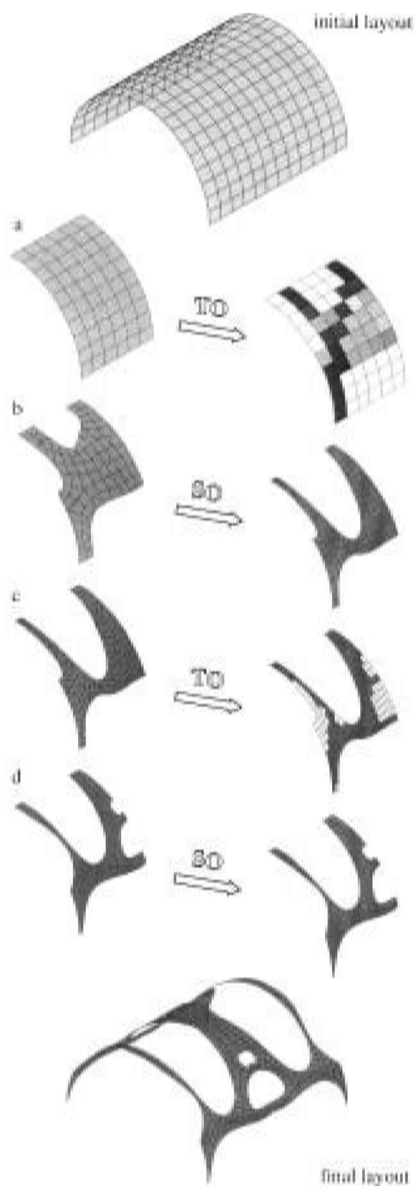


Fig. 12 Combined topology/shape optimization.

analysis, the problem is solved by a sequential quadratic programming (SQP)[22] method in 12 iterations reducing the objective by 46.9%. The final result is shown in Fig. 10c.

Topology Optimization of a Cylindrical Shell

Alternately, material topology optimization and shape optimization are applied to determine the optimum structural layout of a thick cylindrical shell (Fig. 11). The load and support conditions are shown. The objective of the design problem is again maximum stiffness. The available mass is prescribed (average density ratio: 0.3). Because of the symmetry of the problem, only one-quarter of the shell is analyzed and optimized.

The optimization process is shown in Fig. 12. In the first analysis model, a coarse mesh (a) is used to determine roughly the structural layout by conventional material based topology optimization (TO). Based on the artificial material approach with $\mu = 2.5$ an optimality criterion method according to Eq. (16) is used to solve the material distribution problem. A new adapted design model is generated by processing the obtained material distribution. This model can be used either as a new effective design space for a further topology optimization step or as initial structure for shape optimization (SO). To speed up the optimization process, shape optimization with only few design variables is used to determine the optimum shape of the contours for the temporarily fixed topology (b). For shape optimization the material is equally distributed. Consequently, the optimization variables of the shape optimization step are the positions of the control nodes and the one overall density of the smeared material. The shape optimization problem is solved by an SQP[22] method determining the sensitivities analytically. The more detailed contours are obtained by shape optimization. To check whether the fixed topology is still optimal, topology optimization is applied (c). The final layout is determined once again by shape optimization (d).

Conclusions

Based on an adaptive design modeling, material topology and shape optimization methods can be applied alternately or simultaneously to solve generalized shape optimization problems. The design model is successively adapted to the material distribution. A background mesh, which stores the results of the previous optimization step, is introduced. The parametrization of the design model by design patches is used for topology optimization. The parametrization of the contours of the design model by Lagrangian curves or Bézier splines is used for shape optimization.

The proposed procedure provides the possibility to use material-based topology optimization in an efficient, adaptive way. The stability of the procedure was verified by several examples in two[23] and three dimensions. Because the influence of the few parameters of the algorithm on the optimization process, e.g., the threshold value to neglect domains, is transparent, the procedure can be easily controlled. Conventional material topology procedures can be embedded. However, orthotropic material approaches, e.g., rank-2 material, are more advantageous than isotropic approaches, because they are rather independent of mesh distortion and orientation. This was verified for plane-stress problems[9]; in further studies orthotropic approaches for shell problems are to be investigated.

Conventional shape optimization is applied to speed up the adaptive material topology optimization process. With only a few variables, the shape of the inner and outer contours is determined in an intermediate optimization step. Because shape optimization is used within the overall procedure, robust shape optimization procedures are necessary, e.g., adaptive remeshing of the finite element mesh. Moreover, the definition of the optimization variables for the shape optimization step is a crucial subproblem, which is difficult to solve automatically.

Beyond these specific aspects, design adaptivity in structural optimization raises some fundamental questions concerning the numerical approximation of the design function: how to define an approximation error and how to derive refinement indicators. First investigations exist,[5,8] which have to be refined in the future.

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